Fitting Tail Distributions

SAS Working Party

Agenda

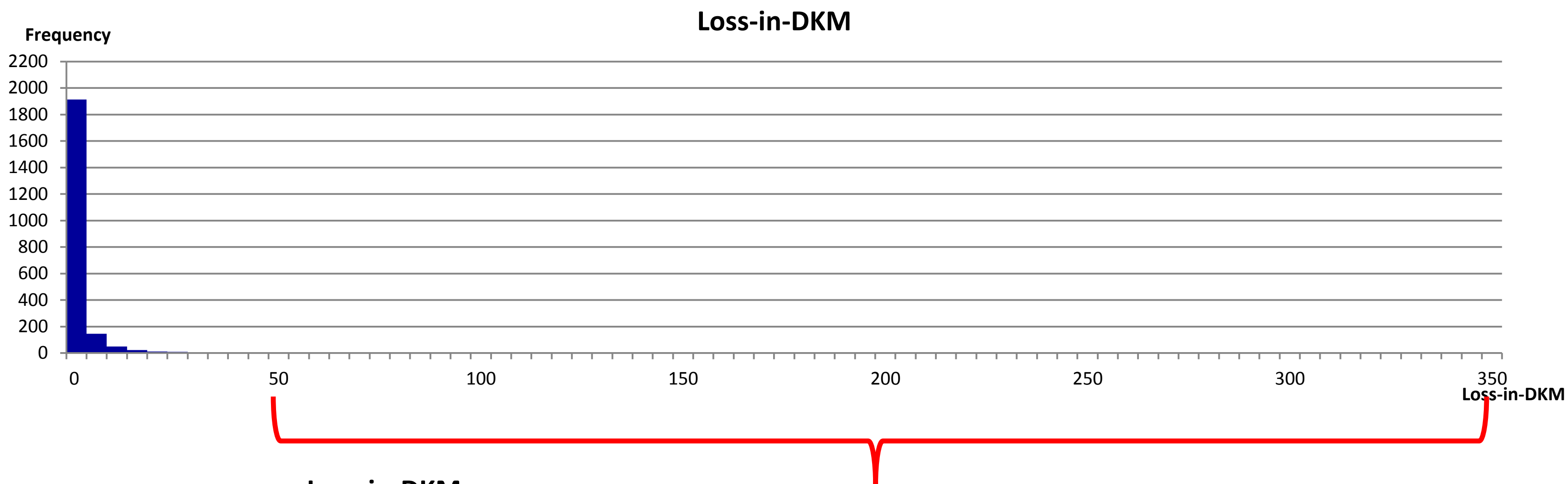
- Why Fit Distributions to Tails?
- Choosing the Right Distribution
- Extreme Value Theory
- Parameter Estimation
- Parameter Uncertainty

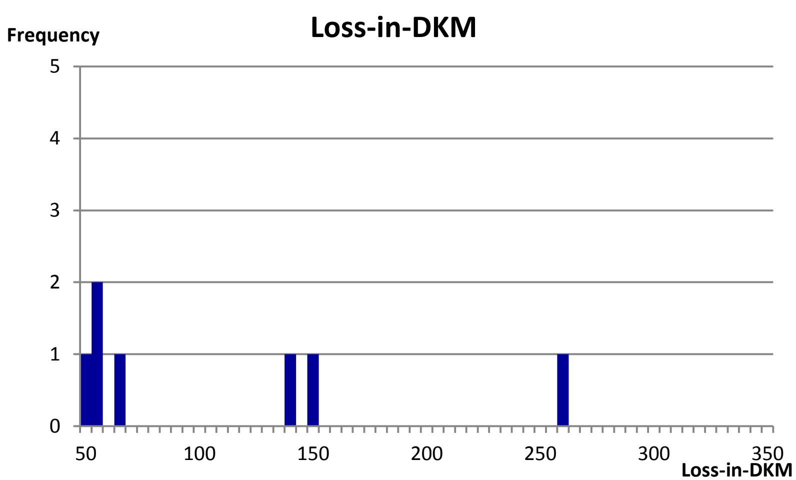
Why????

Why do we need to fit a tail to a distribution – some real life examples

- Pricing non-proportional covers
- Reinsurance requirements and optimisation
- Capital modelling & Enterprise risk management
- Capacity management, including PML estimates

How unreliable the historical data in estimating large loss – A closer look at the scarcity of data



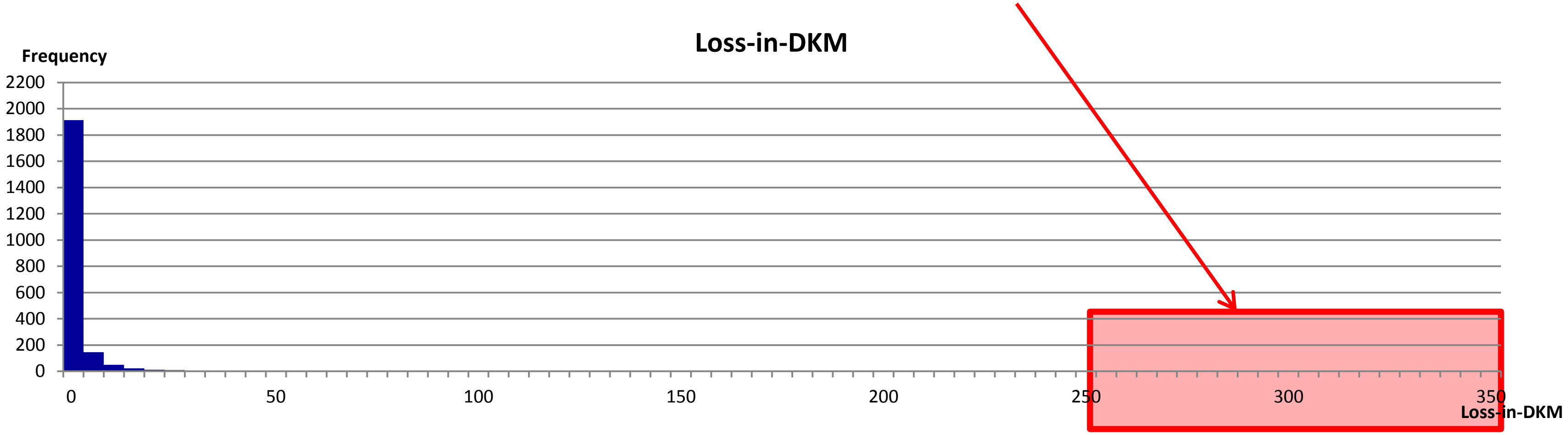


- More than 99.7% of the data centred around smaller losses (losses below $50m)
- Hence, how should we price for the largest and rare losses (the tail)?
- A right model is necessary to estimate the tail (losses above threshold)

XOL Pricing

- For this presentation, we will assume that the actuary is pricing the following excess of loss (XOL) cover:

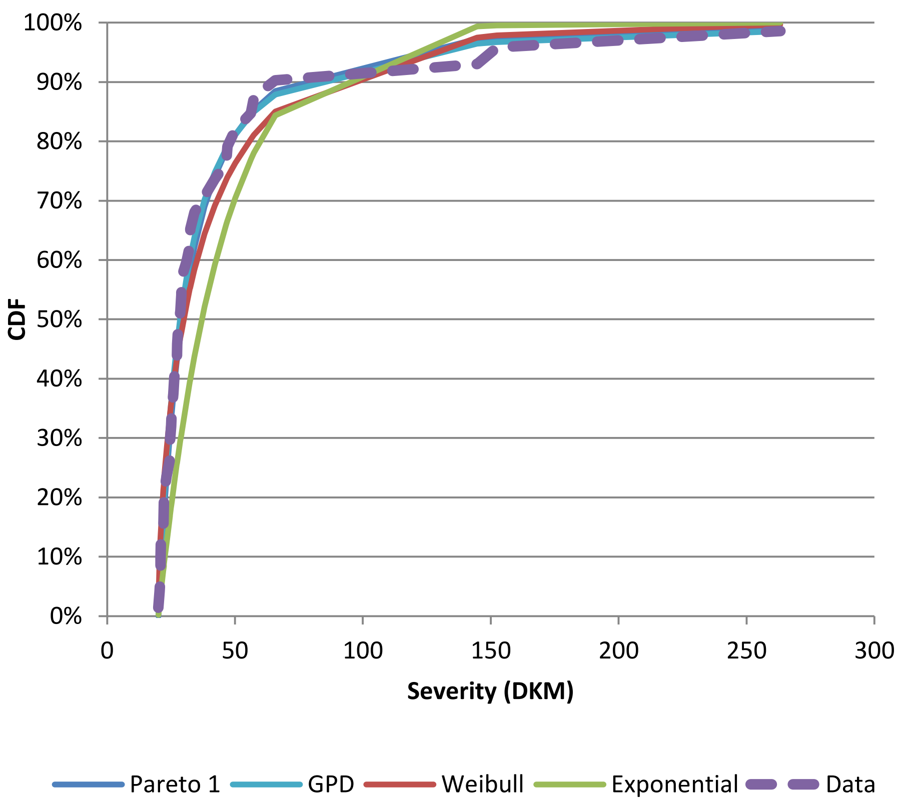
100 DKM excess of 250 DKM

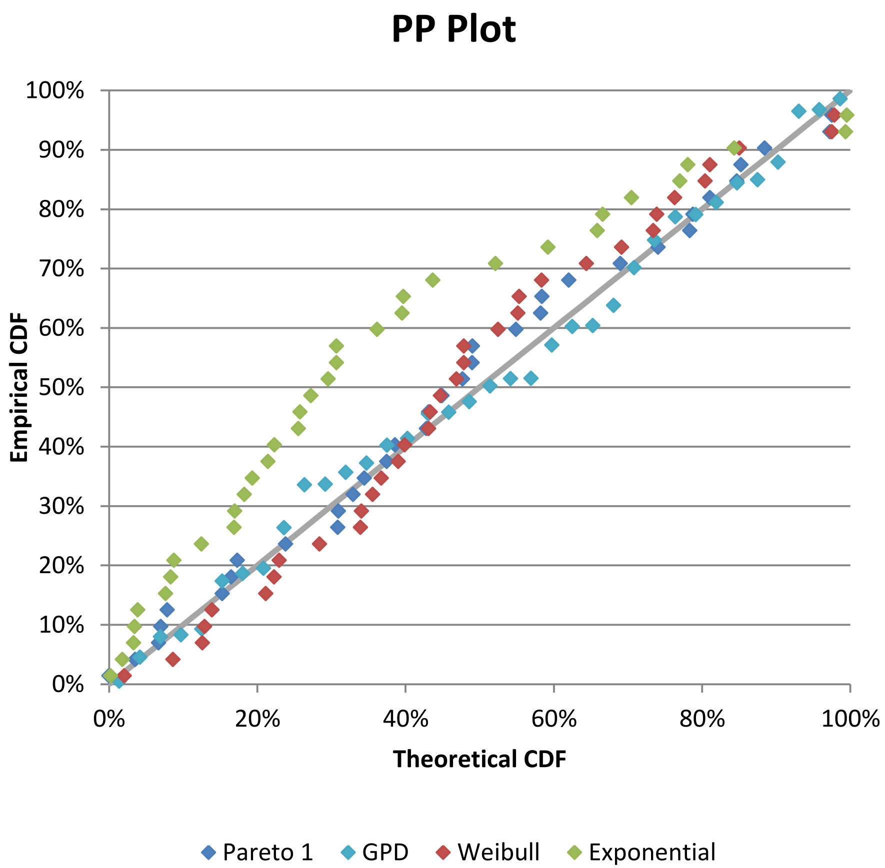


Choosing the Right Distribution

Multiple Distributions Could Be Fitted

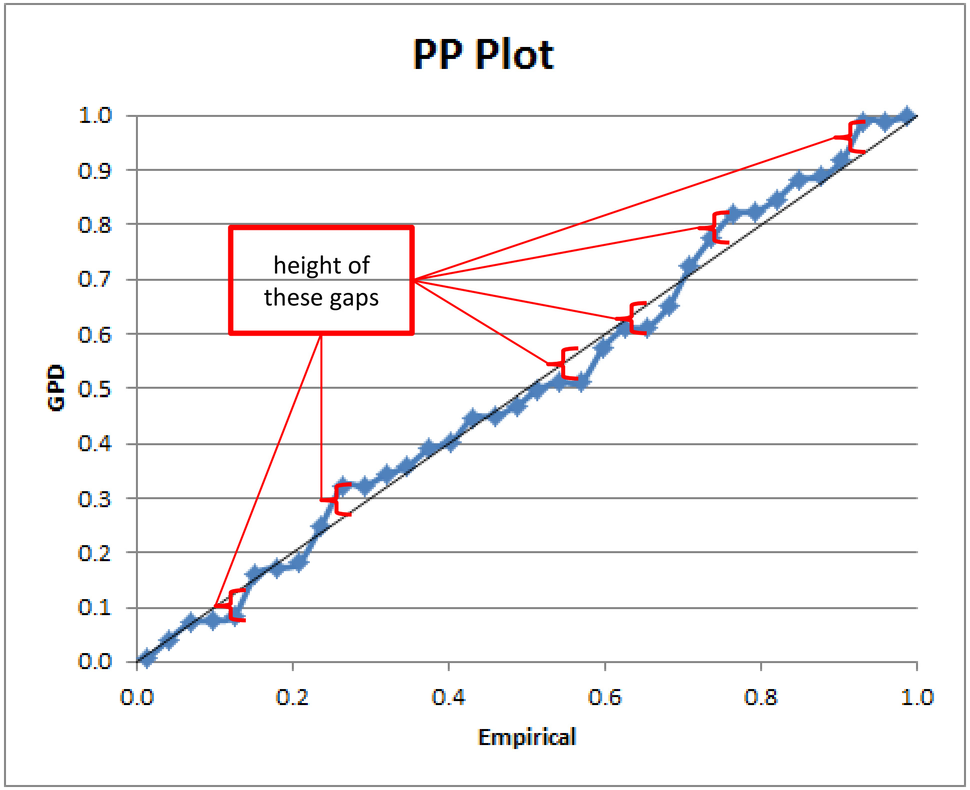
- We've selected 4 possible distributions to fit to this data
- Fitted using MLE parameter estimates





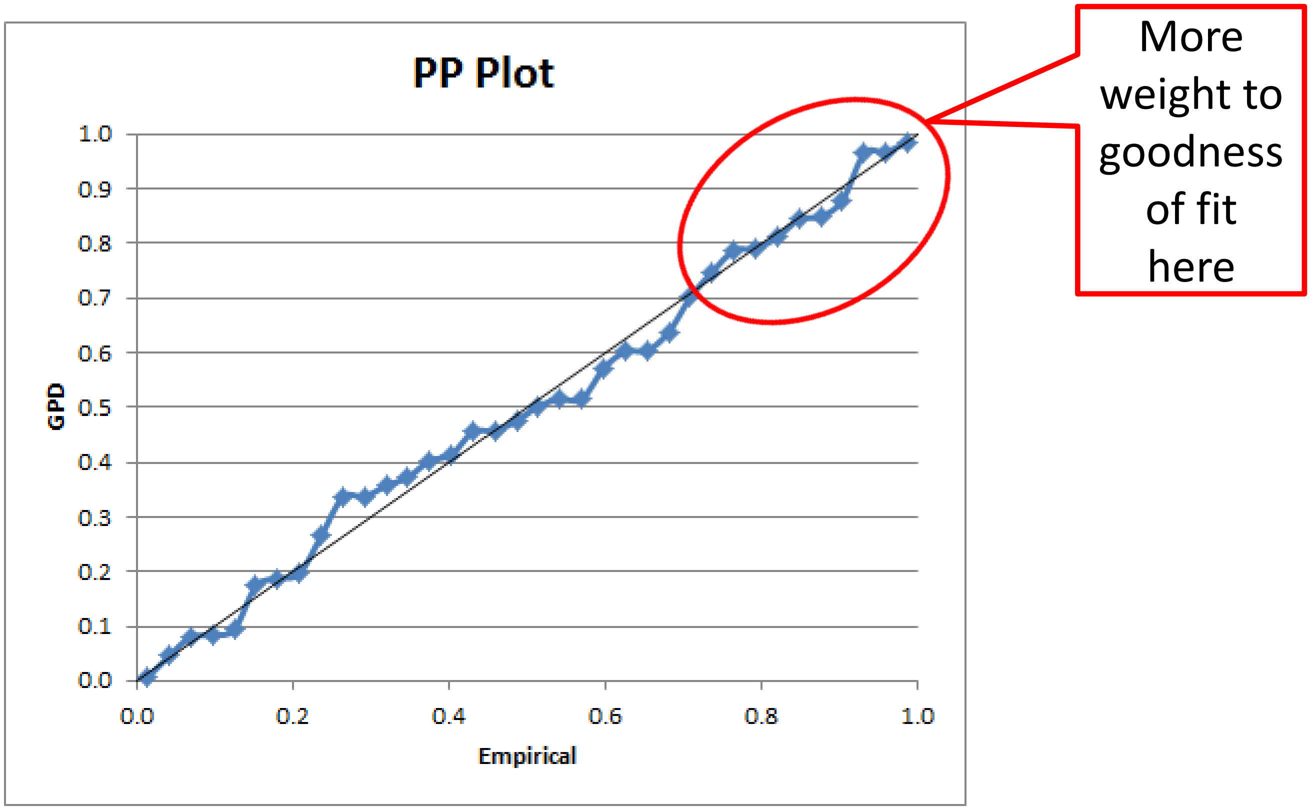
Kolmogorov-Smirnov

- K-S score is largest gap in PP plot



Anderson-Darling

- Similar to Kolmogorov-Smirnov but with greater weight given to fit of higher values



Summary of Distribution Choices

- Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) scores – measures to determine good fit of distributions
- K-S: minimises maximum PP plot residuals
- A-D: gives more weight PP plot residuals in tail of distributions

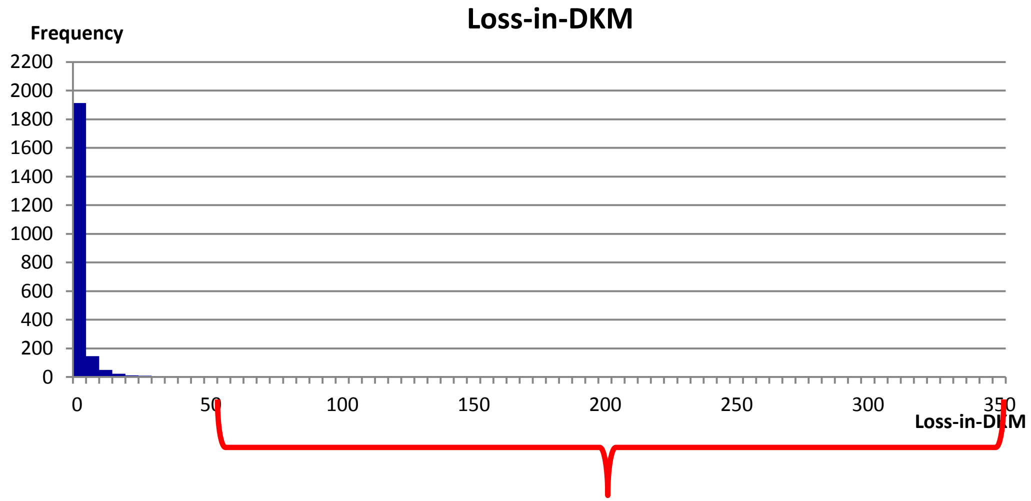
Distribution	K-S score	A-D score	XOL Risk Premium
Pareto 1	0.08	0.50	68.7
Weibull	0.10	0.61	7.4
Exponential	0.26	4.63	0.8
Generalized Pareto	0.07	0.19	43.1

GPD is the best fit for the tail as compared to other distributions

Extreme Value Theory

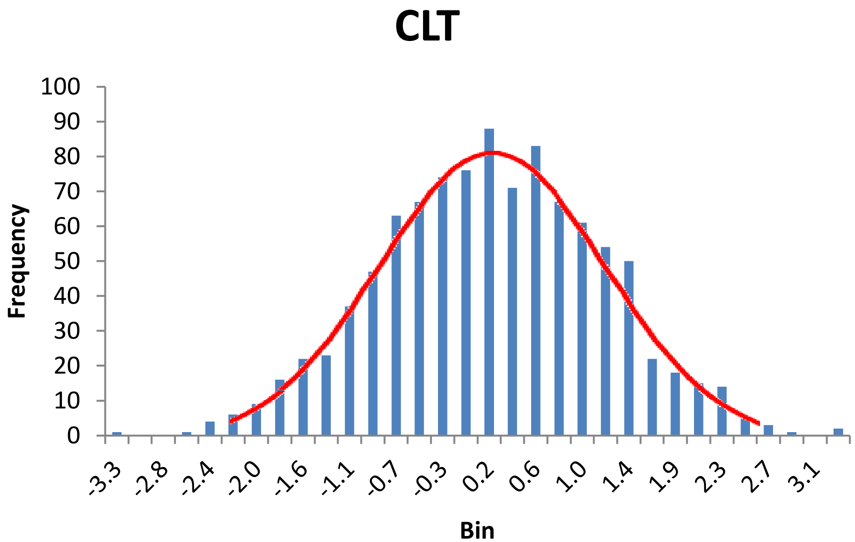
Extreme Value Theory (EVT)

- EVT focuses on **extreme event** data rather than all the data.
  - EVT focuses on the upper tail of the data.
  - EVT studies probabilistic models for the occurrence of rare events.
  - Helps to produce reliable estimates of extreme probabilities



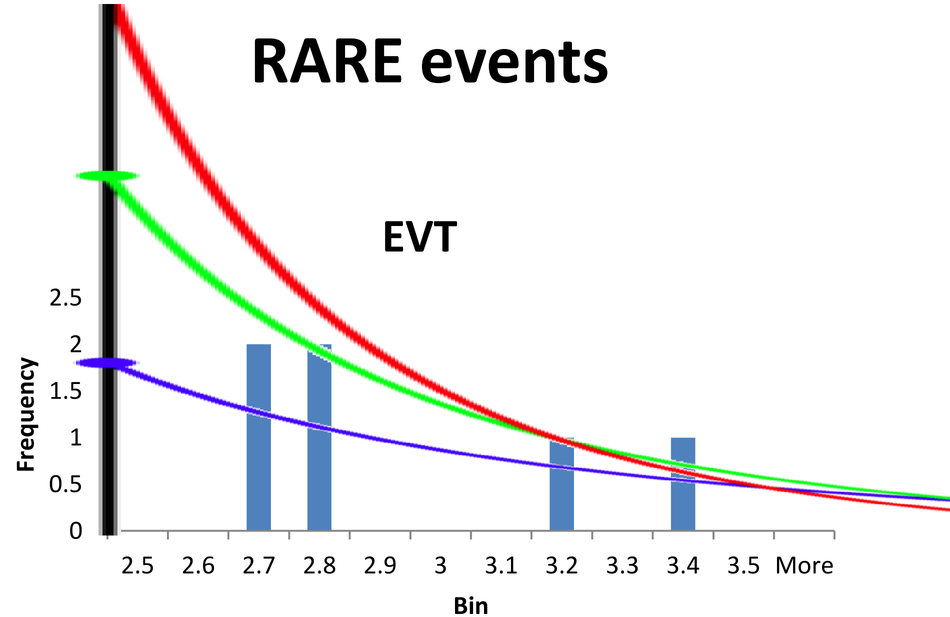
CLT vs. EVT

Central Limit Theorem CLT focuses on AVERAGES



Mean ~ Normal

Extreme Value Theory EVT focuses on EXTREME and RARE events



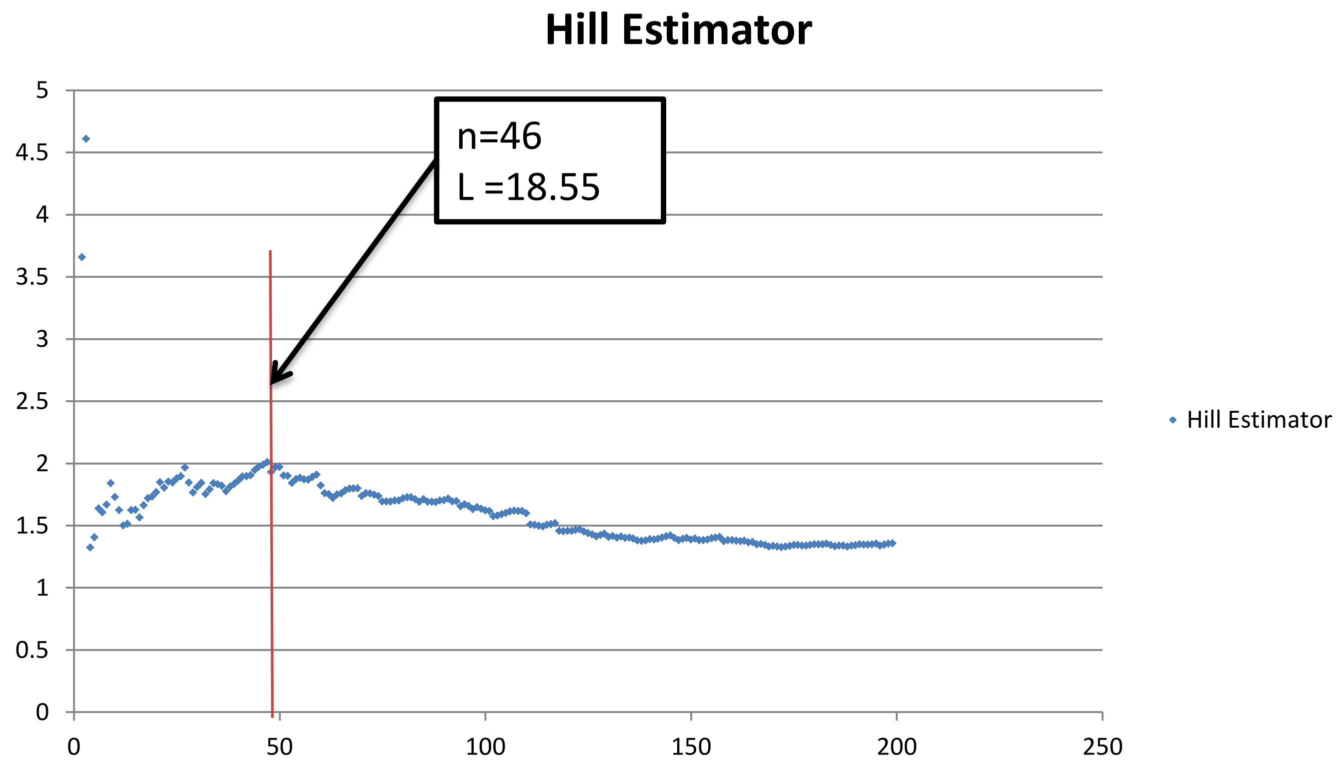
Tail ~ GPD
(with parameters)

Determine threshold "u"

- "u" too high: insufficient data and not robust
- "u" too low: inefficient estimates and may not be GPD
- Method
  - Hill Estimate
  - Mean Excess Plot
  - Sequential Mann-Kendall
  - Ln-Ln Plot
- Choice of u

Hill Estimate

- Looking for threshold such that data is linear to the left

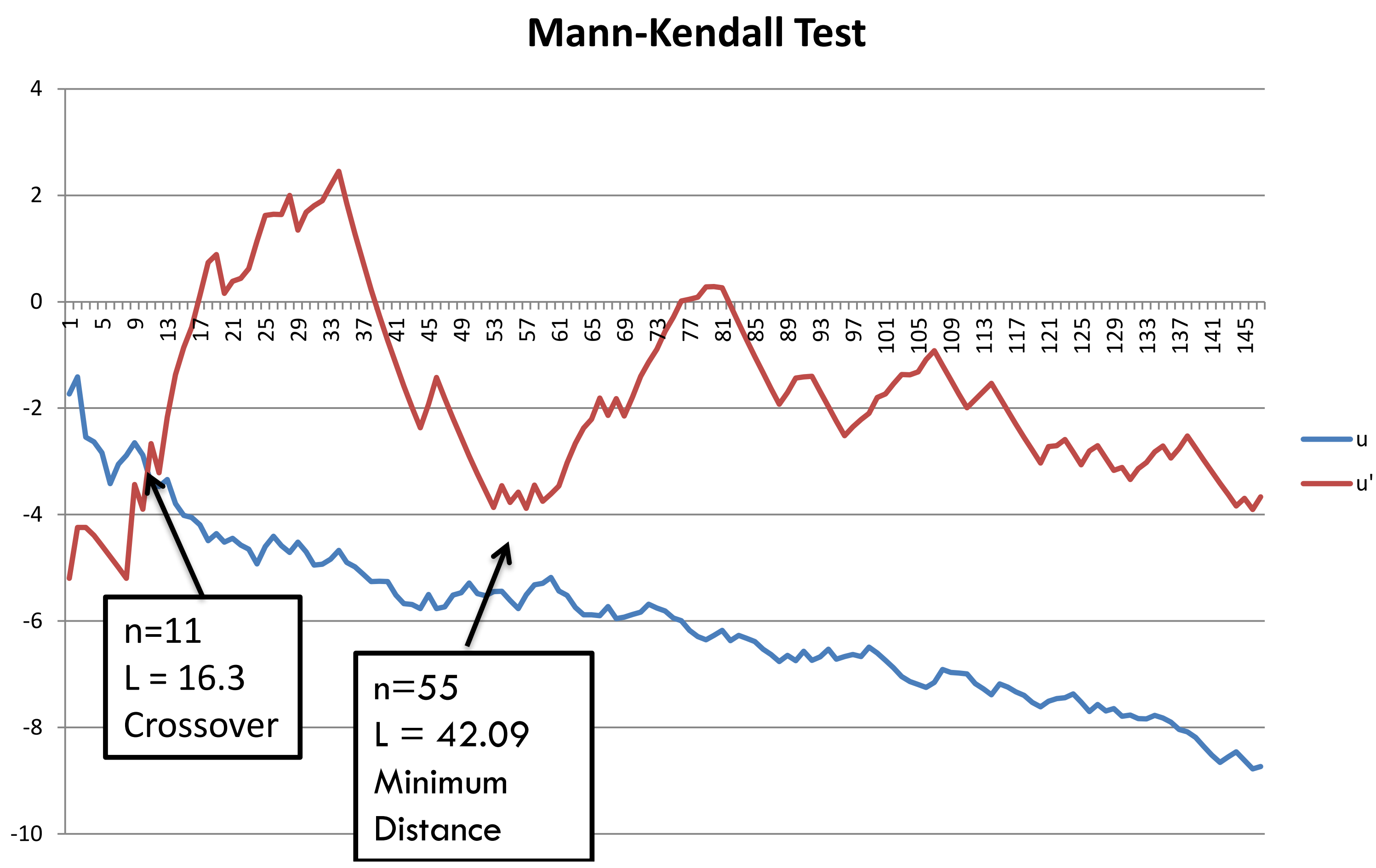


Mean Excess Plot

- Look for threshold such that data is linear to the right

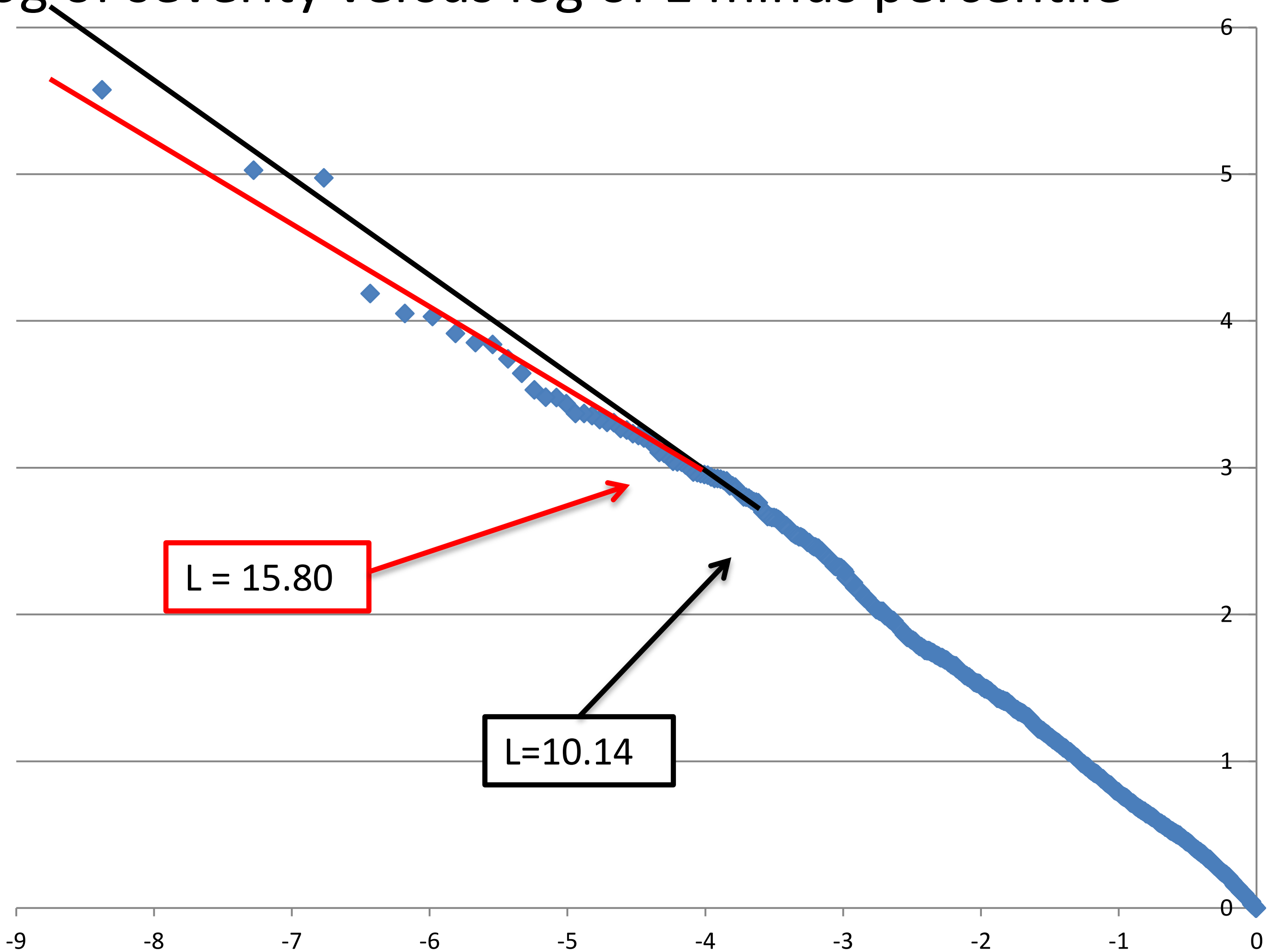
Mann-Kendall Test

- Look for crossover or local minimum distance



Ln-Ln Plot

- Look for threshold such that data is linear to the left
- Log of severity versus log of 1 minus percentile



Conclusion - EVT

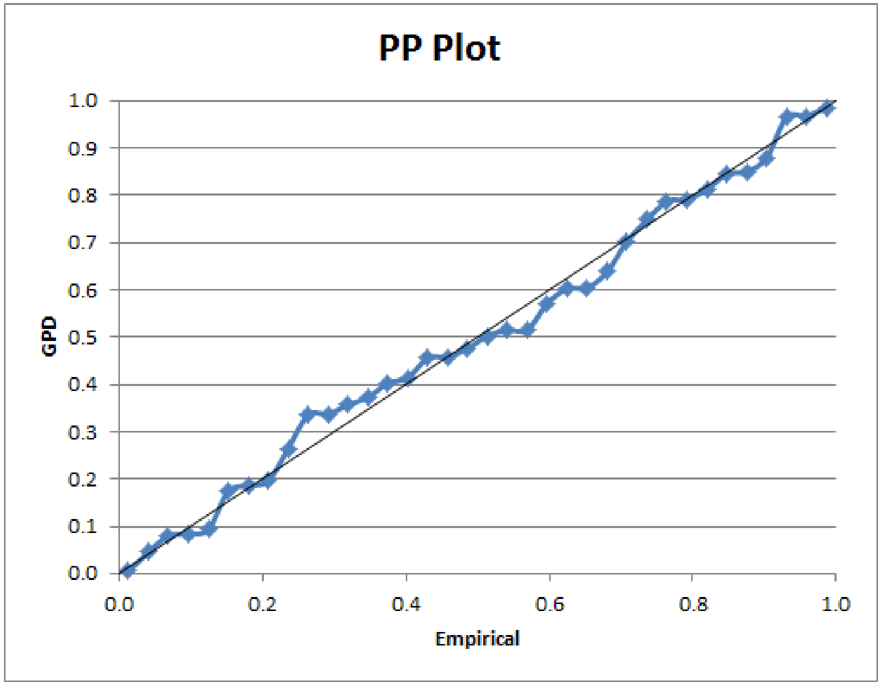
Method	Lower threshold	Upper threshold
Hill Estimate	18.55	n.a.
Mean Excess Plot	17.8	n.a.
Mann-Kendall	16.3	42.9
Ln-Ln Plot	15.80	n.a.
Summary	20	50

- any tail of any GPD is also a GPD
- therefore any threshold above the minimum threshold is OK

Parameter Estimation

MLE

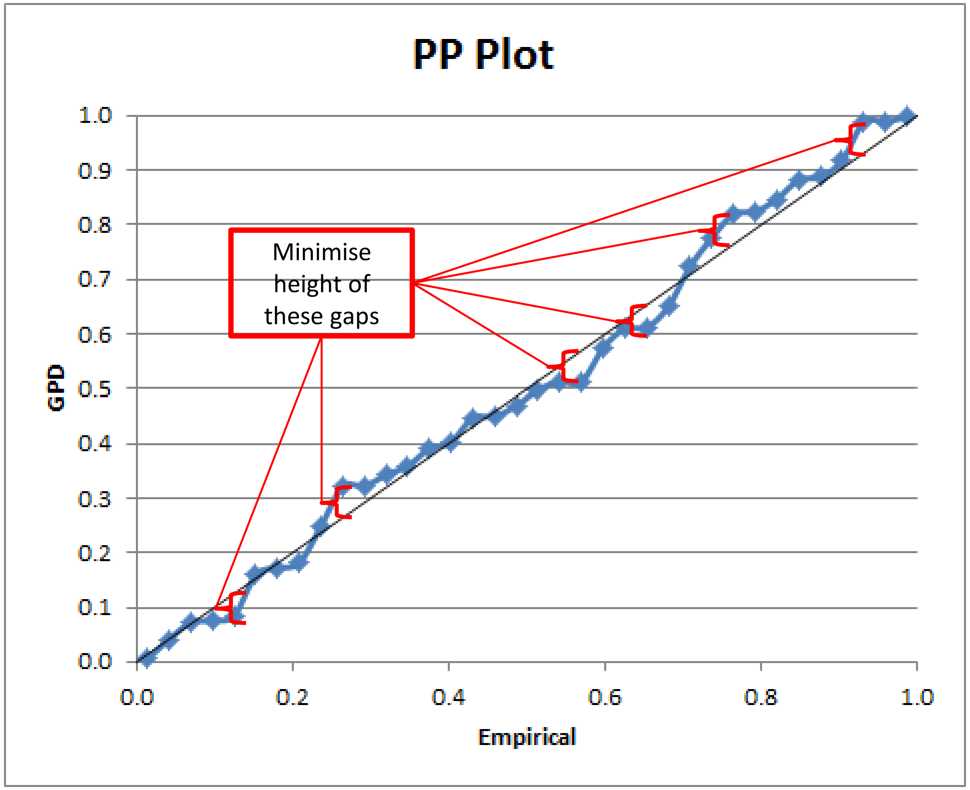
- maximum likelihood – a frequentist approach
- optimises the likelihood function by varying the parameters



- a good fit to Danish fire data

Minimum Kolmogorov-Smirnov

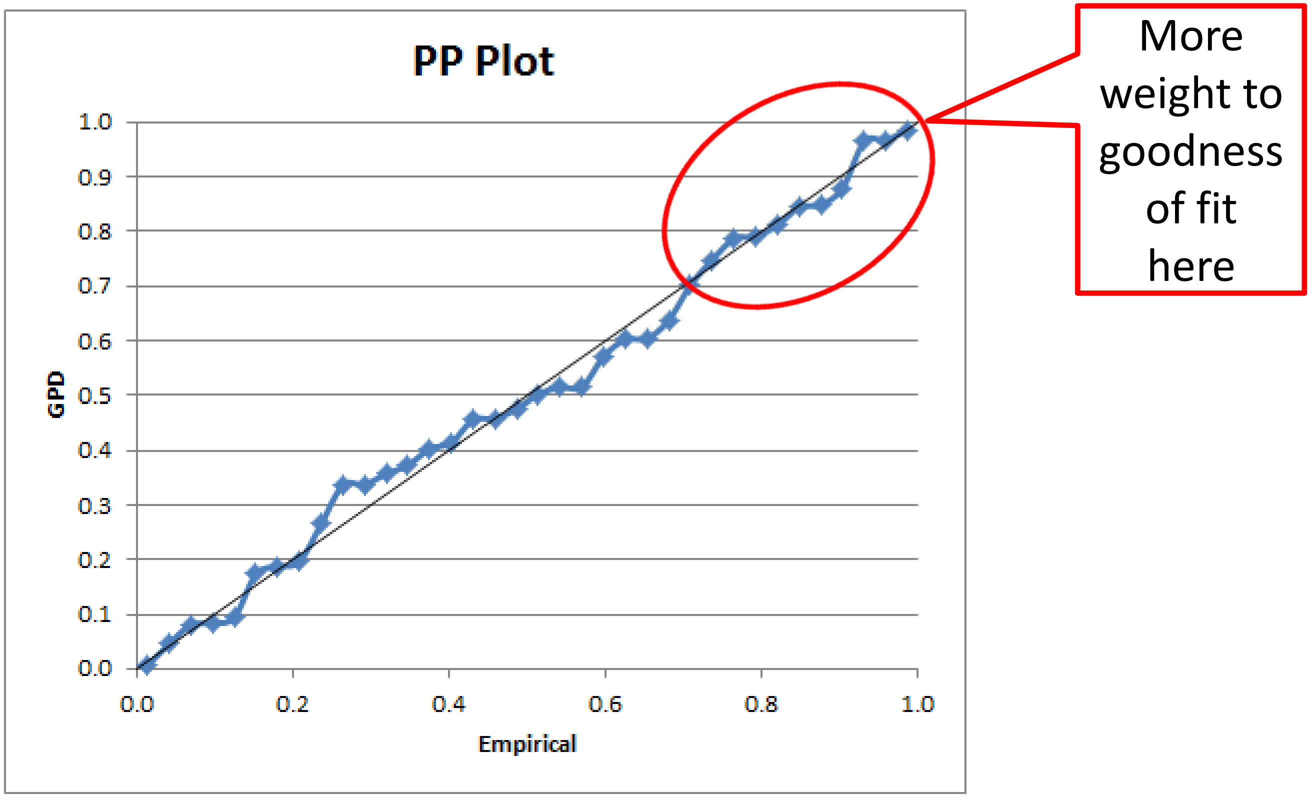
- optimise K-S score by varying the parameters



- better overall fit to GPD

Minimum Anderson-Darling

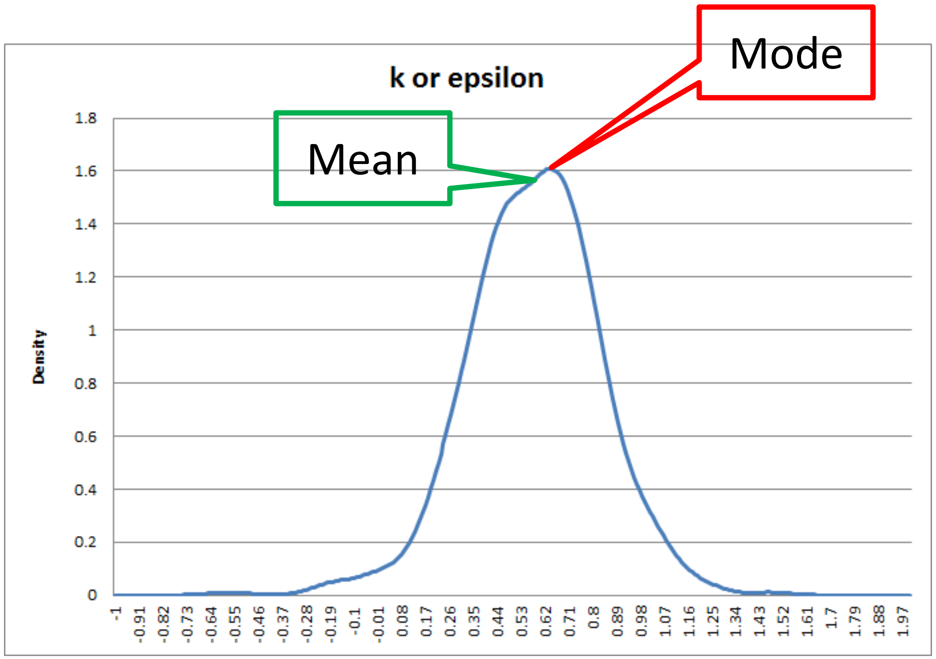
- optimise A-D score by varying the parameters

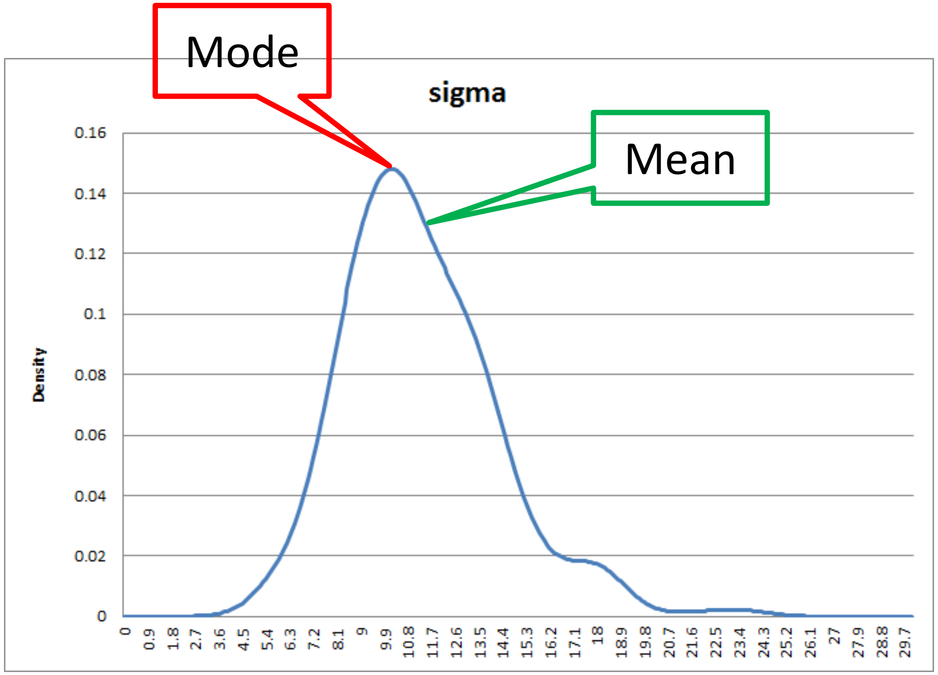


- better fit to higher severities

Bootstrap

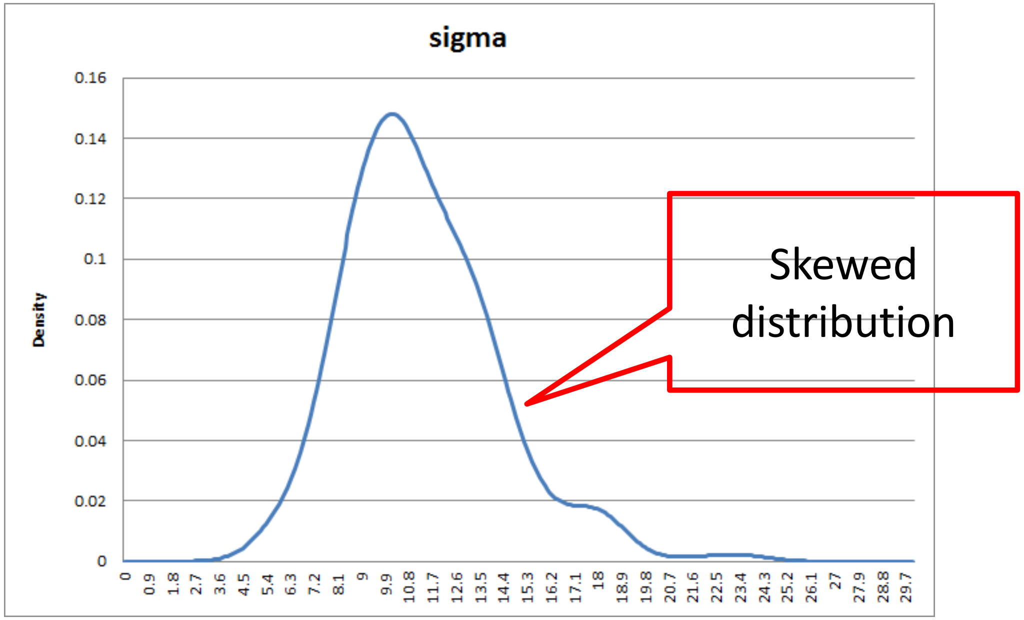
- implementation details discussed later
- demonstrates bias in MLE (i.e. modal) estimates, especially sigma





Correction for MLE Bias

- MLE works best with parameters have symmetric distribution
- Skewed distribution of parameters
- Fix is to
  - estimate bias
  - then adjust



Summary of Parameter Estimates

- all methods

Method	epsilon	sigma	theta	XOL risk premium
Burning Cost				13.3
MLE	0.68	9.7	20	43.1
K-S optimal	0.39	10.9	20	7.8
A-D optimal	0.69	9.6	20	44.3
Bootstrap mean	0.58	11.3	20	46.5
Bootstrap mode	0.64	10.2	20	38.2
MLE bias adjusted	0.86	7.7	20	63.7

- considerable variation between methods
- BEWARE: mean is **higher** than the mode!

Parameter Uncertainty

What is Parameter Uncertainty

- Parameter uncertainty is uncertainty in the values of the parameters of the statistical model.
- Common causes

Change in Parameters over time	Errors in the data	Heterogeneity in the portfolio
Limited amount of data	Uncertainties in the estimate of the losses	

One Method of Bootstrapping

To estimate parameter uncertainty

- **Non Parametric**

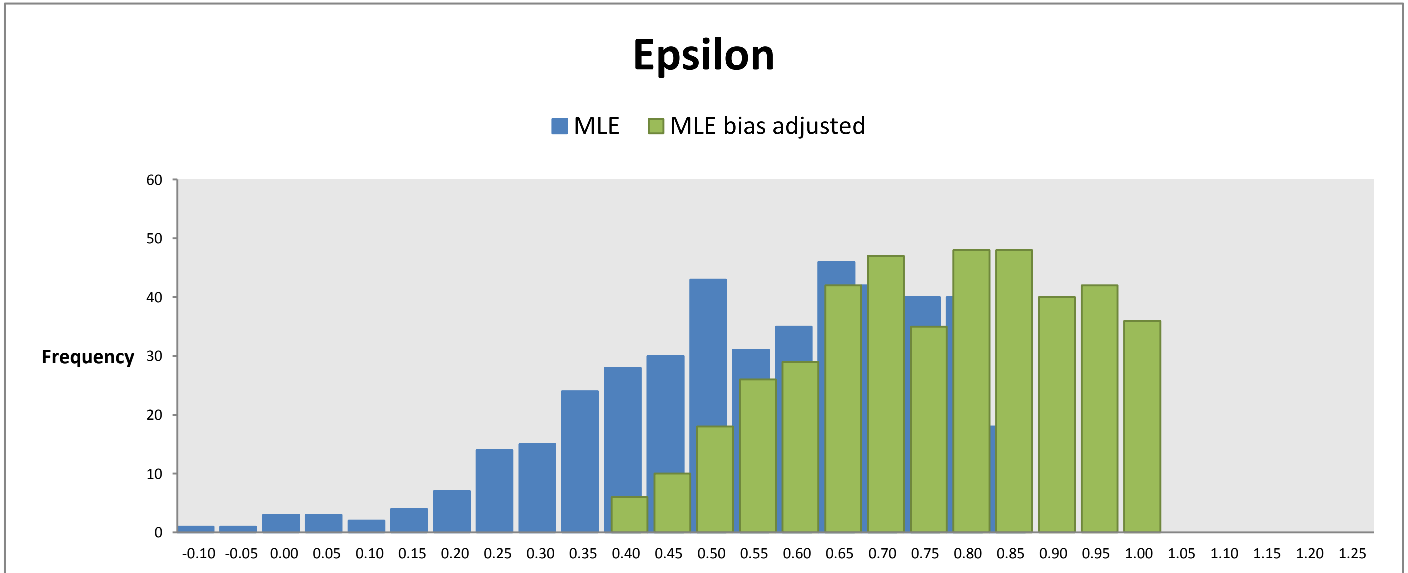
  (suitable when sample size is non trivial)

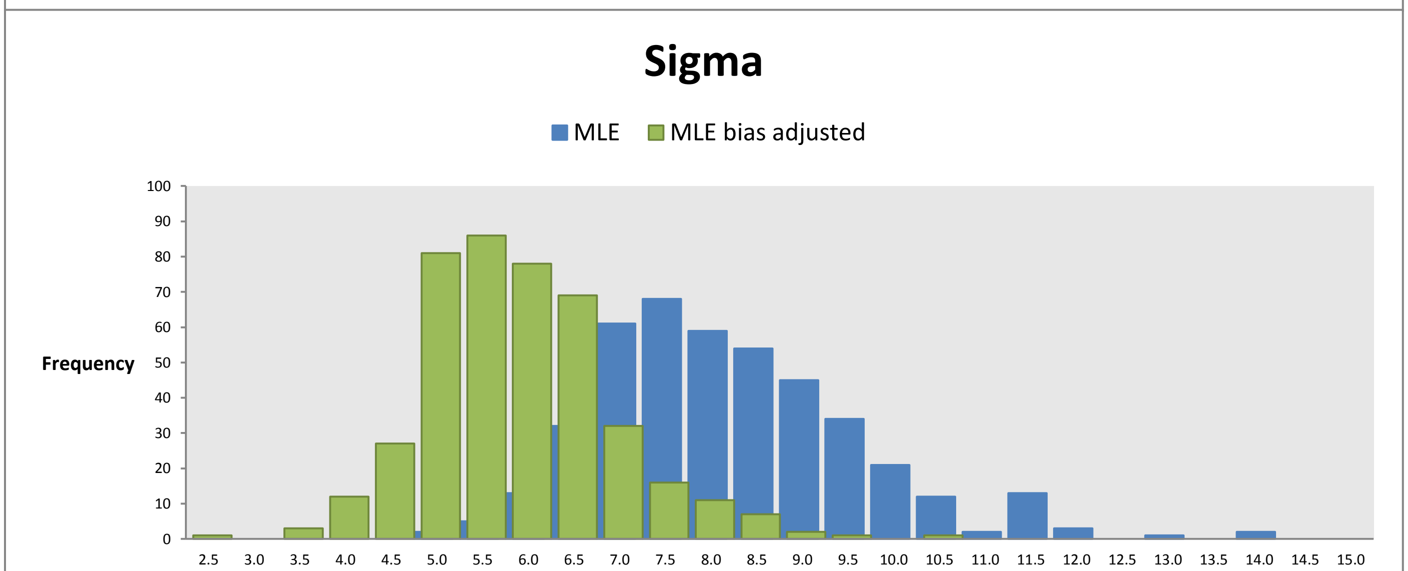
The Non Parametric process

1. Estimate GPD parameters, given the actual historical dataset
2. Sample (with replacement) from the historical dataset, until we get an artificial history with the same number of observations as the actual history
3. Estimate GPD parameters against that artificial history
4. Repeat steps 2&3 many times, storing the parameter estimates each time
5. Show the confidence interval, mean, max and min of each parameter across all the iterations

Final: Compare how the XoL risk premium from step 1 is different to the mean XoL risk premiums across the above iterations

Results of Bootstrap





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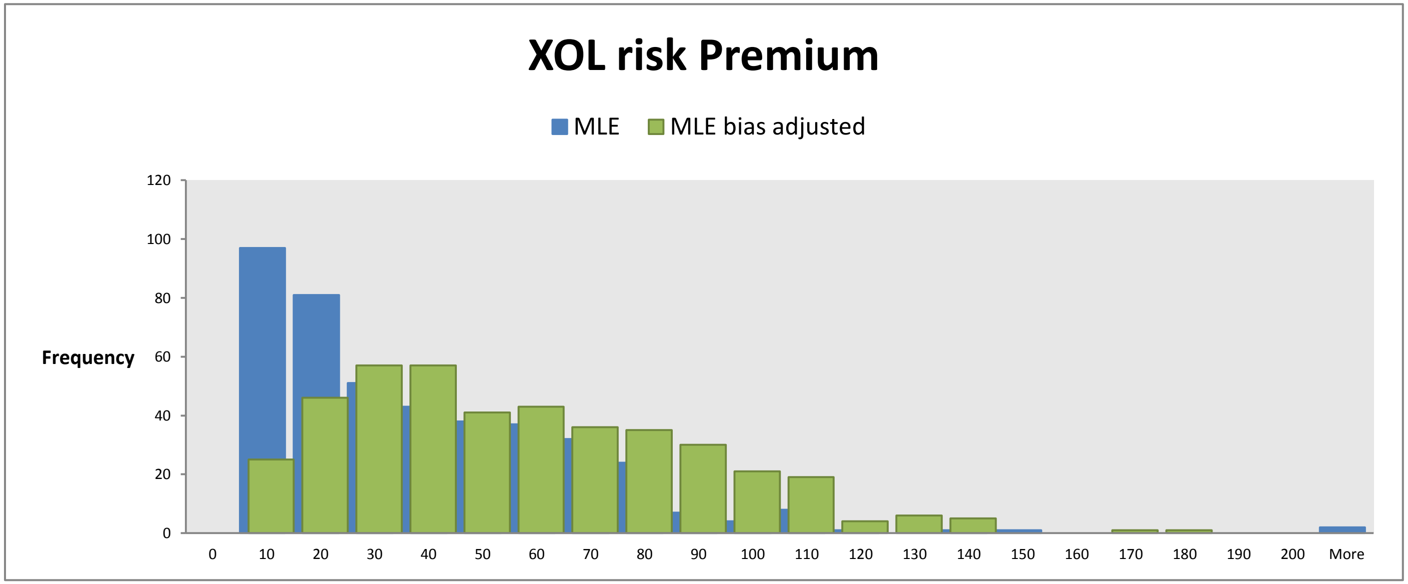
Results of Bootstrap



- MLE underestimates XOL risk premium compared to MLE bias adjusted method
- Parameter estimates by MLE have higher variance

Results

- The following calculations confirms the following ***intuitive*** outcome

- An increase in the estimated XoL reinsurance premium.

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Conclusions

Conclusions

- statistical methods help us make better estimates of rare and extreme events
- using the wrong technique will give you the wrong estimate

- SAS working group plans to run a workshop and provide Excel tools for SAS members