

Valuing Equity-Linked Death Benefits: Option Pricing Without Tears

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ING will take a charge of €0.9 billion – €1.1 billion (U.S. \$1.2 billion - \$1.4 billion) to close its U.S. **variable annuity** business.

ING stopped selling **variable annuities** in 2009.



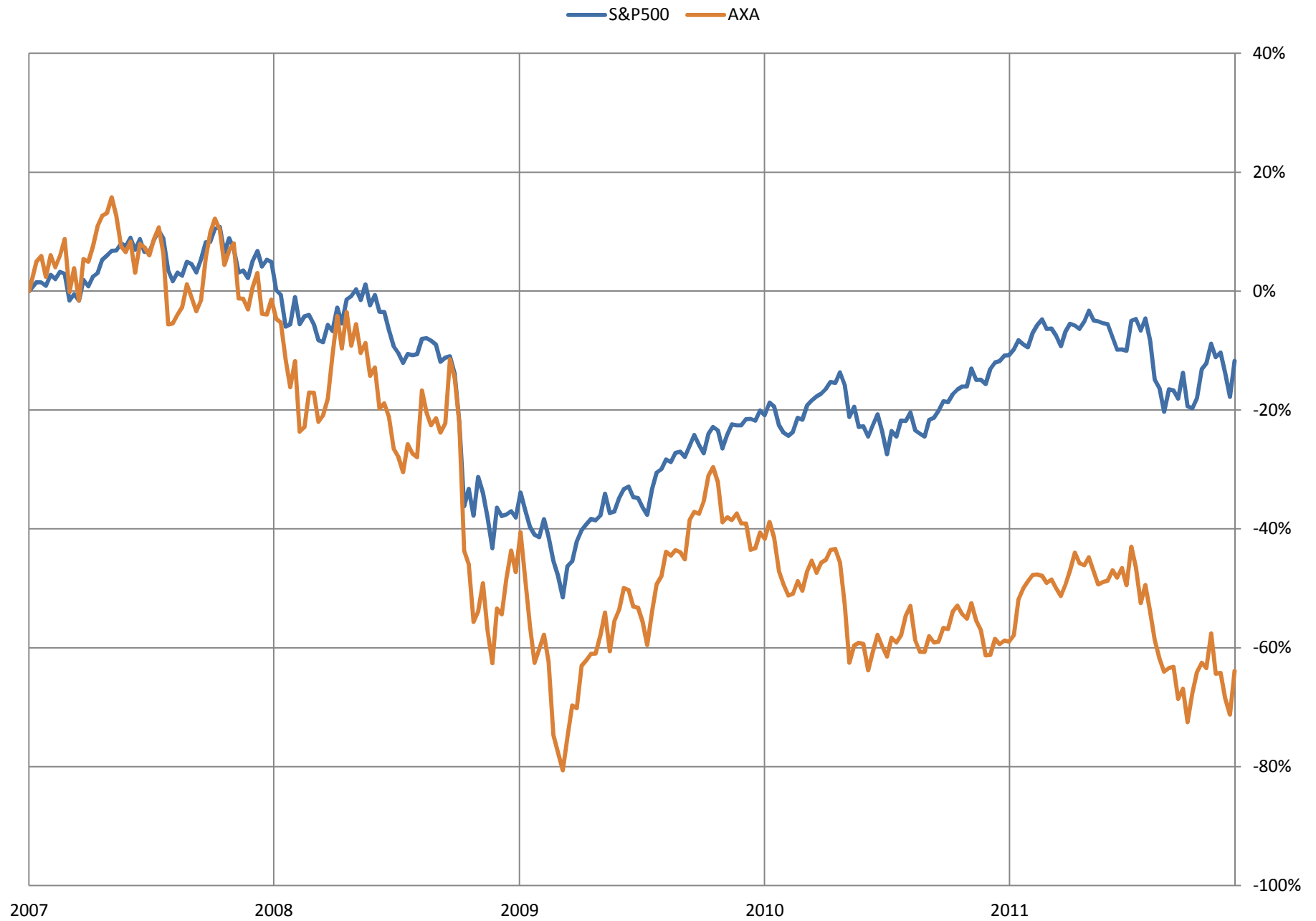
Financial Post, January 29, 2010

“When markets collapsed in September 2008, **Manulife**'s net exposure of guarantees from segregated fund products was [Canadian] \$ 72 billion ...

The company's capital levels ... sank because of the massive stock portfolio associated with the **variable annuities**”

In 2004, Manulife merged with John Hancock, creating the **largest** life insurer in Canada, the **second largest** in the USA, and the **fifth largest** in the world.

In 2004, Manulife decided remove the hedging on the equity positions that it held in its **variable annuity** business. For several years, the decision boosted Manulife's profits. Then ...



“Much of the downturn in AXA's profit outlook comes from variable-annuities hedging. The costs of hedging the annuities – all from its U.S. business – ballooned from 64 million euros in the first half of the year to 123 million euros in the third quarter -- and as much as 450 million euros in the fourth quarter.”

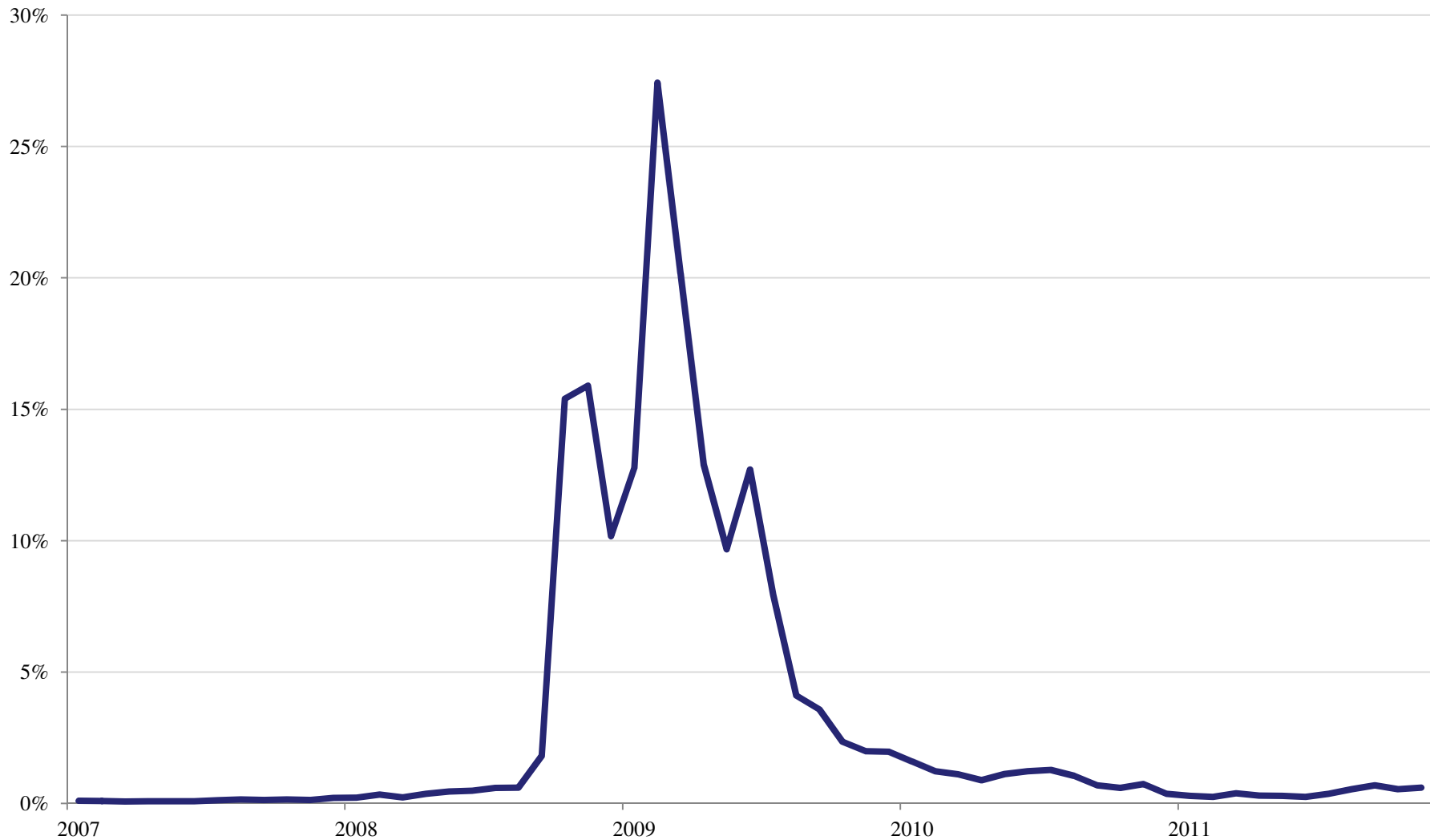
The Wall Street Journal “MarketWatch”, 25 November 2008



“*SNL Financial* reported an industry net loss of \$900.3 million, down from a net income of \$8.9 billion in second-quarter 2009. Primarily **variable annuity** writers were affected ... The largest insurers to see an increase in reserves were **The Hartford**, with an increase in **\$2.7 billion** ...”

TheStreet, Sep 29, 2010

**NUS Risk Management Institute
Credit Rating Initiative
1-Year Probability of Default
Hartford Financial Services (HIG)**



“CIGNA Corporation, on September 3, 2002, announced that it would record an after-tax charge of \$720 million to strengthen reserves relating to reinsurance contracts on variable annuity death-benefit guarantees. Originally, CIGNA Corporation had only set aside \$300 million against these policies and did not use capital markets to hedge the embedded options. In early September 2002, it realized, and then announced, that it had mis-estimated the risk and was forced to add the extra \$720 million to its actuarial reserves”

The Wall Street Journal, Sept. 5, 2002

Table 4.10

Income of Life Insurers, by Year (millions)

Year	Net premium receipts			Total premium receipts	Investment income ¹	Other income ²	Total income
	Life insurance premiums	Annuity considerations	Health insurance premiums				
1911	\$626	\$4	—	\$630	\$182	\$24	\$836
1915	776	6	—	782	241	20	1,043
1920	1,374	7	—	1,381	341	42	1,764
1925	2,340	38	—	2,378	551	89	3,018
1930	3,416	101	—	3,517	891	186	4,594
1935	3,182	491	—	3,673	1,013	386	5,072
1940	3,501	386	—	3,887	1,231	540	5,658
1945	4,589	570	—	5,159	1,445	1,070	7,674
1950	6,249	939	\$1,001	8,189	2,075	1,073	11,337
1955	8,903	1,288	2,355	12,546	2,801	1,197	16,544
1960	11,998	1,341	4,026	17,365	4,304	1,338	23,007
1965	16,083	2,260	6,261	24,604	6,778	1,785	33,167
1970	21,679	3,721	11,367	36,767	10,144	2,143	49,054
1975	29,336	10,165	19,074	58,575	16,488	2,959	78,022
1980	40,829	22,429	29,366	92,624	33,928	4,336	130,888
1985	60,127	53,899	41,837	155,863	67,952	10,212	234,027
1986	66,213	83,712*	44,153	194,078	75,435	12,744	282,257
1987	76,737	88,677	47,549	212,963	82,875	18,460	314,298
1988	73,531	103,278	52,306	229,115	92,042	16,983	338,140
1989	73,290	114,997	56,079	244,366	103,965	18,987	367,318
1990	76,692	129,064	58,254	264,010	111,853	26,337	402,200
1991	79,301	123,590	60,900	263,791	118,984	28,247	411,022
1992	83,868	132,645	65,545	282,058	121,389	23,469	426,916
1993	94,448	156,445	68,658	319,551	124,205	22,594	466,350
1994	98,948	153,019	86,184	338,151	125,999	28,478	492,628
1995	102,766	158,389	90,038	351,193	143,967	32,894	528,054
1996	107,598	178,416	92,183	378,197	152,700	30,190	561,087
1997	115,039	197,529	92,737	405,305	170,713	34,628	610,646
1998	119,897	229,493	94,881	444,271	176,801	42,311	663,383
1999	120,274	270,212	100,049	490,535	186,563	49,830	726,928
2000	130,616	306,693	105,619	542,928	220,862	47,679	811,469
2001	125,314	251,255^	103,413	479,982	203,399	41,068	724,448
2002	134,483	269,296^	108,703	512,482	180,855	40,676	734,013
2003‡	127,320	268,558^	115,827	511,705	179,744	35,558	727,007
2004‡	139,691	276,677^	125,752	542,120	186,827	27,863	756,810
2005‡	142,261	277,117^	118,267	537,645	206,859	34,521	779,024
2006‡	149,223	302,727^	141,198	593,149	239,669	50,779	883,597
2007‡	142,661	314,225^	151,462	608,348	267,394	74,624	950,366
2008‡	147,182	328,135^	165,034	640,350	260,123	40,166	940,638

Source: ACLI tabulations of National Association of Insurance Commissioners (NAIC) data, used by permission.

Income of U.S. Life Insurers

Source: Data from *Life Insurers' Fact Book*

(US\$ **million**)

Year	Life Insurance Premiums	Annuity Considerations	Health Insurance Premiums
1950	6,249	939	1,001
1960	11,998	1,341	4,026
1970	21,679	3,721	11,367
1980	40,829	22,429	29,366
1986	66,213	83,712	44,153
1990	76,692	129,064	58,254
2000	130,616	306,693	105,619
2005	142,261	277,117	118,267
2008	147,182	328,135	165,034

Annuities

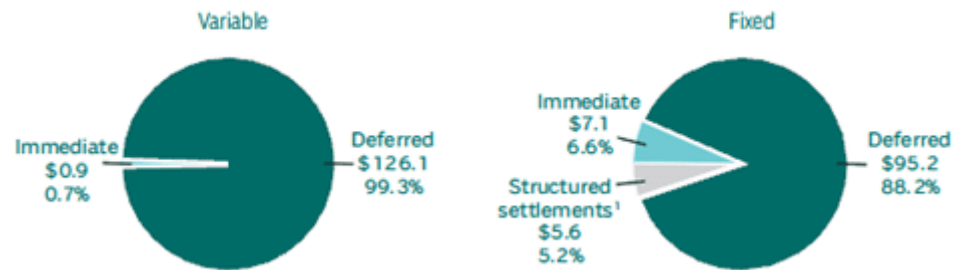
Deferred Annuities -- Accumulation

Immediate Annuities -- Payout

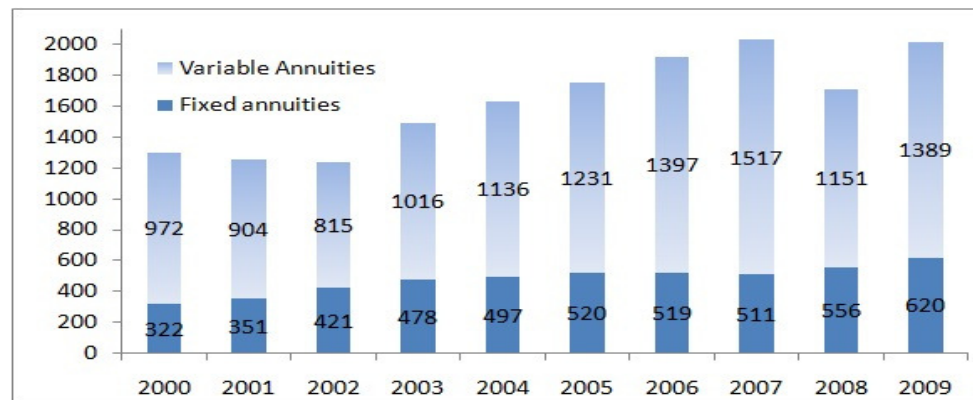
Fixed Annuities

Variable Annuities

Equity-Indexed Annuities



Annuity Sales by Product Type, 2009 (\$ billion)



Deferred Annuity Assets, 2000-2009 (\$ billion, year-end)

Source: The U.S. Individual Annuity Yearbook - 2009 by LIMRA

What are Variable Annuities (VA)?

Variable Annuities

= Investment Funds (Mutual Funds)

+

Rider(s) : **G**uaranteed **M**inimum **B**enefits

I. **G****M****D****B**

Death

II. **G****M****A****B**

Accumulation

III. **G****M****I****B**

Income

IV. **G****M****W****B**

Withdrawal

The basic Variable Annuity product:

Allows policyholders to invest proceeds into a variety of investment funds.

Deposits grow on a tax-deferred basis. There a 10% federal-tax penalty on withdrawals before age 59½. Withdrawals after 59½ are taxed as income (not capital gains).

Monthly asset management, expense, and risk charges are deducted by the insurer, usually as a (fixed) % of the prevailing account balance.

Sold mostly to ages 50+ and frequently rolled over within the VA market after 5 to 7 years. Roll-overs, referred to as 1035 exchanges, represents a significant portion of “new” sales.

Valuing Guaranteed Minimum Death Benefits (GMDB)

Consider a Variable Annuity (VA) policy issued to a life aged x .

Let $S(t)$ be the investment fund value at time t .

Let $T(x)$ denote the time-until-death random variable.

In terms of traditional actuarial notation,

$$\Pr[T(x) > t] = {}_t p_x,$$

$$\Pr[T(x) \leq t] = {}_t q_x,$$

and the probability density function of $T(x)$ is

$${}_t p_x \mu_{x+t}.$$

Let $T(x)$ denote the time-until-death random variable for a person age x .

Let $S(t)$ be the price of a stock or stock fund at time t , $t \geq 0$.

Consider the GMDB payoff

$$\text{Max}(S(T(x)), K),$$

where K is the guaranteed amount, e.g., $K = S(0)$.

One way to value this guarantee is to calculate

$$E[e^{-\delta T(x)} \text{Max}(S(T(x)), K)],$$

where the expectation is taken with respect to an appropriate probability distribution and δ is a valuation force of interest.

Let $T(x)$ denote the time-until-death random variable for a person age x .

Let $S(t)$ be the price of a stock or stock fund at time t , $t \geq 0$.

Stock-dependent death benefit:

$$b(S(T(x)))$$

Example: $b(s) = \text{Max}(s, K) = s + (K - s)_+$

Example: $b(s) = (K - s)_+$

Problem: Evaluate

$$E[e^{-\delta T(x)} b(S(T(x)))],$$

where δ is a “valuation” force of interest and the expectation is taken with respect to an *appropriate* probability measure.

By conditioning on the time of death,

$$\begin{aligned} & E[e^{-\delta T(x)} b(S(T(x)))] \\ &= E[E[e^{-\delta T(x)} b(S(T(x))) \mid T(x)]] \\ &= \int_0^{\infty} E[e^{-\delta t} b(S(t)) \mid T(x) = t] f_{T(x)}(t) dt \\ &= \int_0^{\infty} E[e^{-\delta t} b(S(t))] f_{T(x)}(t) dt \end{aligned}$$

if $T(x)$ is independent of $\{S(t)\}$.

So we want to calculate

$$\int_0^{\infty} E[e^{-\delta t} b(S(t))] f_{T(x)}(t) dt.$$

If

$$f_{T(x)}(t) = \sum_j c_j f_{T_j}(t) ,$$

then

$$\begin{aligned} & \int_0^{\infty} E[e^{-\delta t} b(S(t))] f_{T(x)}(t) dt \\ &= \sum_j c_j \int_0^{\infty} E[e^{-\delta t} b(S(t))] f_{T_j}(t) dt \\ &= \sum_j c_j E[e^{-\delta T_j} b(S(T_j))]. \end{aligned}$$

We know

$$f_{T(x)}(t) \approx \sum_j c_j f_{T_j}(t) ,$$

with $\{T_j\}$ being *exponential* random variables. Thus, the problem of valuing a stock-dependent death benefit becomes finding

$$E[e^{-\delta T} b(S(T))],$$

where T is an *exponential* random variable independent of $\{S(t)\}$. It turns out that this is an elementary calculus exercise for geometric Brownian motion $\{S(t)\}$.

Let $S(t) = S(0)e^{\mu t + \sigma Z(t)}$, $t \geq 0$,

where $\{Z(t)\}$ is a standard Brownian motion.

Let T be an independent exponential r. v. with mean $1/\lambda$. Then,

$$E[e^{-\delta T} b(S(T), \text{Max}\{S(t); 0 \leq t \leq T\})]$$

$$= \frac{2\lambda}{\sigma^2} \int_0^\infty \left[\int_{-\infty}^m b(S(0)e^x, S(0)e^m) e^{-\alpha x} dx \right] e^{-(\beta - \alpha)m} dm,$$

where $\alpha < 0$ and $\beta > 0$ are the roots of

$$\frac{1}{2}\sigma^2 x^2 + \mu x - (\lambda + \delta) = 0.$$

$$E[e^{-\delta T} b(S(T), \text{Max}\{S(t); 0 \leq t \leq T\})]$$

$$= \frac{2\lambda}{\sigma^2} \int_0^\infty \left[\int_{-\infty}^m b(S(0)e^x, S(0)e^m) e^{-\alpha x} dx \right] e^{-(\beta-\alpha)m} dm.$$

$$b(s, u) = (K - s)_+ \quad \text{put option}$$

$$E[e^{-\delta T} (K - S(T))_+ | S(0) \geq K]$$

$$= \frac{2\lambda}{\sigma^2} \int_0^\infty \left[\int_{-\infty}^m (K - S(0)e^x)_+ e^{-\alpha x} dx \right] e^{-(\beta-\alpha)m} dm$$

$$= \frac{2\lambda}{\sigma^2} \int_0^\infty \left[\int_{-\infty}^{\ln[K/S(0)]} \right] = \frac{2\lambda}{\sigma^2} \left[\int_{-\infty}^{\ln[K/S(0)]} \right] \int_0^\infty e^{-(\beta-\alpha)m} dm$$

Put option

$$E[e^{-\delta T} (K - S(T))_+ \mid S(0) \geq K]$$

$$= \frac{2\lambda}{\sigma^2} \frac{K}{-\alpha(1-\alpha)} \left[\frac{S(0)}{K} \right]^\alpha \frac{1}{\beta - \alpha},$$

where $S(t) = S(0)e^{\mu t + \sigma Z(t)}$, T is an exponential random variable independent of $\{S(t)\}$ with mean $1/\lambda$, and $\alpha < 0$ and $\beta > 0$ are the roots of

$$\frac{1}{2}\sigma^2 x^2 + \mu x - (\lambda + \delta) = 0.$$

Put option

$$E[e^{-\delta T} (K - S(T))_+ \mid S(0) \geq K]$$

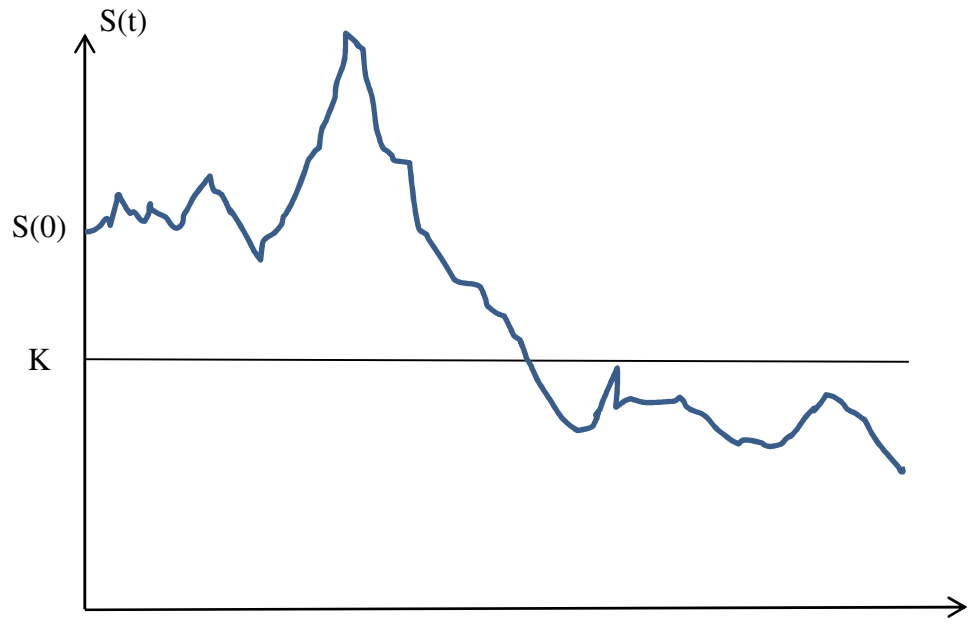
$$= \frac{2\lambda}{\sigma^2} \frac{K}{-\alpha(1-\alpha)} \left[\frac{S(0)}{K} \right]^\alpha \frac{1}{\beta - \alpha},$$

$$= \left[\frac{S(0)}{K} \right]^\alpha \times E[e^{-\delta T} (K - S(T))_+ \mid S(0) = K]$$

Out-of-the-money put

$$= \left[\frac{S(0)}{K} \right]^\alpha \times \text{At-the-money put}$$

What is the meaning of $\left[\frac{S(0)}{K} \right]^\alpha$?



Hint: The street address of the Society of Actuaries is
475 North **Martingale** Road

$$E[e^{-\delta T} b(S(T), \text{Max}\{S(t); 0 \leq t \leq T\})]$$

$$b(s, u) = (cu - s)_+ \text{ where } 0 < c \leq 1$$

Fractional floating strike lookback put option

$$E[e^{-\delta T} (c \times \text{Max}\{S(t); 0 \leq t \leq T\} - S(T))_+]$$

$$= c^{1-\alpha} \frac{\lambda}{\lambda + \delta} \frac{\beta}{(1-\alpha)(\beta-1)} S(0)$$

$$= c^{1-\alpha} E[e^{-\delta T} (\text{Max}\{S(t); 0 \leq t \leq T\} - S(T))],$$

which is not true if T is a fixed time.

Consider a fixed time t . The price of the payoff $b(S(t))$ is an expected discounted value,

$$E[e^{-\delta t} b(S(t))],$$

if δ is the *risk-free* force of interest r and the probability measure is *risk-neutral*.

Now, suppose that we insist on using the actual probability measure; then we need to use a *risk-adjusted* force of interest δ .

If $b(S(t)) = [S(t) - K]_+$, the payoff of a *call* option, then $\delta > r$.

If $b(S(t)) = [K - S(t)]_+$, the payoff of a *put* option, then $\delta < r$ and probably δ is *negative*.

Why?

$S(t) = S(0)e^{\mu t + \sigma Z(t)}$, $\{Z(t)\}$ a standard Brownian motion. The stock pays no dividends.

SoA Exam MFE (CAS Exam 3F), Section 20.4

“Sharpe Ratios”:

$$\frac{\mu - \frac{1}{2}\sigma^2 - r}{\sigma} = \frac{\text{"}\delta\text{"} - r}{\Omega \times \sigma}$$

where Ω is the *option elasticity* with respect to the stock price. $\Omega < 0$ for put options.

Both Ω and “ δ ” are functions of time and stock price.

May 2007 Society of Actuaries Exam MFE

- 8.** Let $S(t)$ denote the price at time t of a stock that pays no dividends. The Black-Scholes framework holds. Consider a European call option with exercise date T , $T > 0$, and exercise price $S(0)e^{rT}$, where r is the continuously compounded risk-free interest rate.

You are given:

- (i) $S(0) = \$100$
- (ii) $T = 10$
- (iii) $\text{Var}[\ln S(t)] = 0.4t, \quad t > 0.$

Determine the price of the call option.

$S(t) = S(0)e^{\mu t + \sigma Z(t)}$, $\{Z(t)\}$ a standard Brownian motion. The stock pays no dividends.

Consider the guarantee at time t

$$\text{Max}[S(t), S(0)e^{rt}].$$

(Here, “ratchet” rate = risk-free interest rate.)

By Black & Scholes, the price at time 0 is

$$S(0) \times 2\Phi(1/2\sigma\sqrt{t}),$$

where $\Phi(\cdot)$ is the standard normal c.d.f.

The price does not depend on μ or r .

For large t , it is close to $2 \times S(0)$.